Supersymmetry

CERN/Fermilab Hadron Collider Physics Summer School

Fermilab, August 18-20, 2008

Stephen P. Martin

Northern Illinois University

Based in part on “A Supersymmetry Primer”, `hep-ph/9709356v4` (revised June 2006).

Lecture 1: Motivation and Introduction to Supersymmetry

- Motivation: The Hierarchy Problem
- Supermultiplets
- Particle content of the Minimal Supersymmetric Standard Model (MSSM)
- Need for “soft” breaking of supersymmetry
- The Wess-Zumino Model
- The supersymmetry algebra
- The superpotential

There are good reasons to believe that the next discoveries beyond the presently known Standard Model will involve **supersymmetry (SUSY)**.

Some of them are:

- A possible cold dark matter particle
- A light Higgs boson, in agreement with precision electroweak constraints
- Unification of gauge couplings
- Mathematical beauty

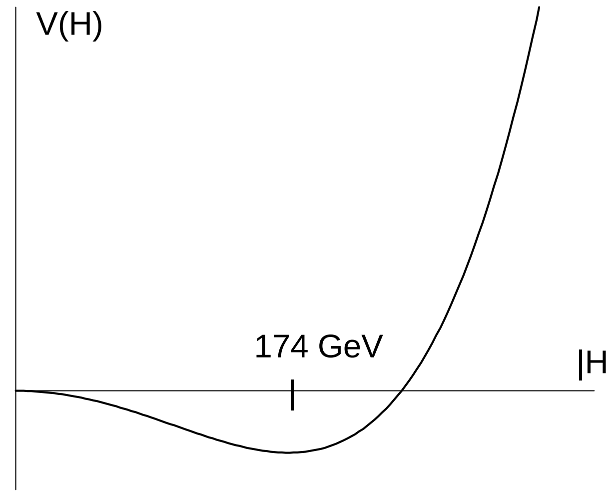
However, they are all insignificant compared to the one really good reason to suspect that supersymmetry is real:

- **The Hierarchy Problem**

The Hierarchy Problem

Consider the potential for H, the complex scalar field that is the electrically neutral part of the Standard Model Higgs field:

$$V(H) = m_H^2 |H|^2 + \frac{\lambda}{2}|H|^4$$



For electroweak symmetry breaking to agree with the experimental m_Z, we need:

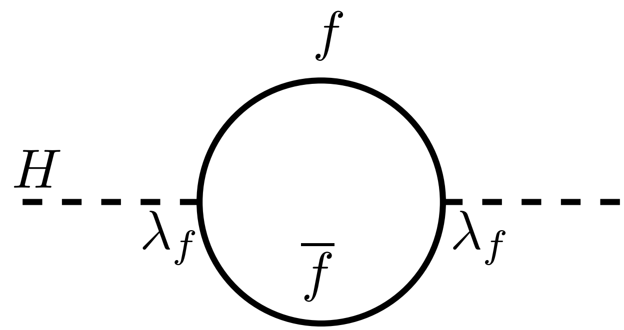
$$\langle H \rangle = \sqrt{-m_H^2/\lambda} \approx 175 \text{ GeV}$$

The requirement of unitarity in the scattering of Higgs bosons and longitudinal W bosons tells us that λ is not much larger than 1. Therefore,

$$-(\text{few hundred GeV})^2 \lesssim m_H^2 < 0.$$

However, this appears fine-tuned (in other words, incredibly and mysteriously lucky!) when we consider the likely size of quantum corrections to m_H^2.

Contributions to m_H^2 from a Dirac fermion loop:



The correction to the Higgs squared mass parameter from this loop diagram is:

$$\Delta m_H^2 = \frac{\lambda_f^2}{16\pi^2} \left[-2M_{\mathrm{UV}}^2 + 6m_f^2 \ln\left(M_{\mathrm{UV}}/m_f\right) + \ldots \right]$$

where λ_f is the coupling of the fermion to the Higgs field H.

M_{UV} should be interpreted as the ultraviolet cutoff scale(s) at which new physics enters to cut off the loop integrations.

So m_H^2 is sensitive to the **largest** mass scales in the theory.

For example, some people believe that String Theory is responsible for modifying the high energy behavior of physics, making the theory finite. Compared to field theory, string theory modifies the Feynman integrations over Euclidean momenta:

$$\int d^4p\,[\ldots] \;\rightarrow\; \int d^4p\, e^{-p^2/M^2_{\rm string}}\,[\ldots]$$

Using this, one obtains from each Dirac fermion one-loop diagram:

$$\Delta m_H^2 \;\sim\; -\frac{\lambda_f^2}{8\pi^2} M^2_{\rm string} + \ldots$$

A typical guess is that $M_{\rm string}$ is comparable to $M_{\rm Planck} \approx 2.4 \times 10^{18}$ GeV.

This makes it difficult to explain how m_H^2 could be so small, after incorporating these relatively huge corrections.

The Hierarchy Problem

We already know:

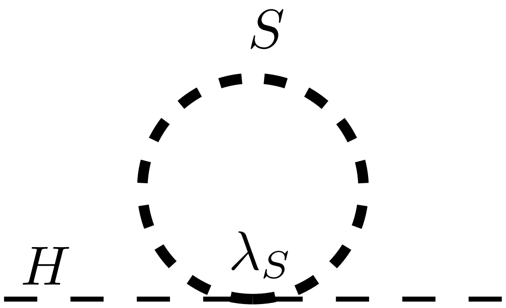
$$\frac{|m_H^2|}{M_{\rm Planck}^2} \lesssim 10^{-32}$$

Why should this number be so small, if individual radiative corrections Δm_H^2 can be of order $M_{\rm Planck}^2$ or $M_{\rm string}^2$, multiplied by loop factors?

This applies even if String Theory is wrong and some other unspecified effects modify physics at $M_{\rm Planck}$, or any other very large mass scale, to make the loop integrals converge.

An incredible coincidence seems to be required to make the corrections to the Higgs squared mass cancel to give a much smaller number.

Scalar loops give a “quadratically divergent” contribution to the Higgs squared mass also. Suppose S is some heavy complex scalar particle that couples to the Higgs.

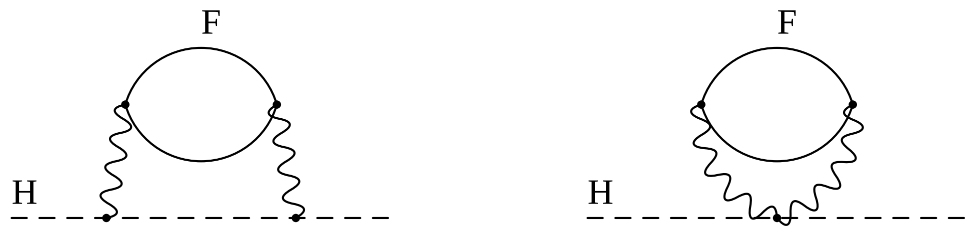


$$\Delta m_H^2 = \frac{\lambda_S}{16\pi^2}\left[M_{\mathrm{UV}}^2 - 2m_S^2 \ln\left(M_{\mathrm{UV}}/m_S\right) + \ldots\right]$$

(Note that the coefficient of the M_{UV}^2 term from a scalar loop has the **opposite sign** of the fermion loop.)

In dimensional regularization, the terms proportional to M_{UV}^2 do not occur. One could adopt dimensional regularization (although it seems unphysical for this purpose), and also assume that the Higgs does not couple directly to *any* heavy particles. But there is still a problem. . .

Indirect couplings of the Higgs to heavy particles still give a problem:



(a) (b)

Here F is any heavy fermion that shares gauge quantum numbers with the Higgs boson. Its mass m_F does not come from the Higgs boson and can be arbitrarily large. From these diagrams one finds (x is a group-theory factor):

$$\Delta m_H^2 = x\left(\frac{g^2}{16\pi^2}\right)^2 \left[kM_{\rm UV}^2 + 48m_F^2 \ln(M_{\rm UV}/m_F) + \ldots\right]$$

Here k depends on the choice of cutoff procedure (and is 0 in dimensional regularization). However, the contribution proportional to m_F^2 is always present.

More generally, *any* indirect communication between the Higgs boson and very heavy particles, or very high-mass phenomena in general, can give an unreasonably large contribution to m_H^2.

The systematic cancellation of loop corrections to the Higgs mass squared requires the type of conspiracy that is better known to physicists as a **symmetry**.

Fermion loops and boson loops gave contributions with opposite signs:

$$\Delta m_H^2 = -\frac{\lambda_f^2}{16\pi^2}(2M_{\rm UV}^2) + \ldots \qquad \text{(Dirac fermion)}$$

$$\Delta m_H^2 = +\frac{\lambda_S}{16\pi^2}M_{\rm UV}^2 + \ldots \qquad \text{(complex scalar)}$$

So we need a **SUPERSYMMETRY** = a symmetry between fermions and bosons.

It turns out that this makes the cancellation not only possible, but automatic.

More on this later, but first, an historical analogy...

An analogy: Coulomb self-energy correction to the electron's mass

H. Murayama, `hep-ph/0002232`

If the electron is really pointlike, the classical electrostatic contribution to its energy is infinite.

Model the electron as a solid sphere of uniform charge density and radius R:

$$\Delta E_{\rm Coulomb} = \frac{3e^2}{20\pi\epsilon_0 R}$$

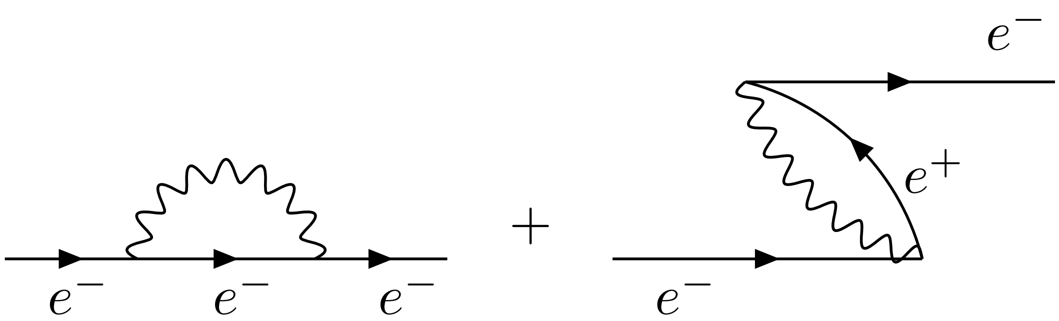
Interpreting this as a correction $\Delta m_e = \Delta E_{\rm Coulomb}/c^2$ to the electron mass:

$$m_{e,\rm physical} = m_{e,\rm bare} + (1\,{\rm MeV}/c^2)\left(\frac{0.9\times 10^{-17}\,{\rm meters}}{R}\right).$$

A divergence arises if we try to take $R \to 0$. Naively, we might expect $R \gtrsim 10^{-17}$ meters, to avoid having to tune the bare electron mass to better than 1%, for example:

$$0.511\,{\rm MeV}/c^2 = -100.000\,{\rm MeV}/c^2 + 100.511\,{\rm MeV}/c^2.$$

However, there is another important quantum mechanical contribution:



The virtual positron effect cancels most of the Coulomb contribution, leaving:

$$m_{e,\text{physical}} = m_{e,\text{bare}} \left[1 + \frac{3\alpha}{4\pi} \ln\left(\frac{\hbar / m_e c}{R} \right) + \dots \right]$$

with $\hbar/m_e c = 3.9 \times 10^{-13}$ meters. Even if R is as small as the Planck length 1.6×10^{-35} meters, where quantum gravity effects become dominant, this is only a 9% correction.

The existence of a "partner" particle for the electron, the positron, is responsible for eliminating the dangerously huge contribution to its mass.

The “reason” for the positron’s existence can be understood from a **symmetry**, namely the Poincaré invariance of Einstein’s relativity when applied to the quantum theory of electrons and photons (QED).

If we did not yet know about relativity or the positron, we would have had three options:

- **Assume that the electron is not point-like, and has structure at a measurable size R.**
- **Assume that the electron is (nearly?) pointlike, and there is a mysterious fine-tuning between the bare mass and the Coulomb correction to it.**
- **Predict that the electron’s symmetry “partner”, the positron, must exist.**

Today we know that the last option is the correct one.

Supersymmetry

A SUSY transformation turns a boson state into a fermion state, and vice versa. So the operator Q that generates such transformations acts, schematically, like:

$$Q|\text{Boson}\rangle = |\text{Fermion}\rangle; \qquad\qquad Q|\text{Fermion}\rangle = |\text{Boson}\rangle$$

This means that Q must be an anticommuting spinor. This is an intrinsically complex object, so $Q^\dagger$ is also a distinct symmetry generator:

$$Q^\dagger|\text{Boson}\rangle = |\text{Fermion}\rangle; \qquad\qquad Q^\dagger|\text{Fermion}\rangle = |\text{Boson}\rangle$$

The possible forms for such theories are highly restricted by the Haag-Lopuszanski-Sohnius extension of the Coleman-Mandula Theorem. In a 4-dimensional theory with chiral fermions (like the Standard Model) and non-trivial scattering, then Q carries spin-$1/2$ with L helicity, and $Q^\dagger$ has spin-$1/2$ with R helicity, and they must satisfy…

The Supersymmetry Algebra

$$\begin{aligned}
\{Q, Q^\dagger\} &= P^\mu \\
\{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0 \\
[P^\mu, Q] &= [P^\mu, Q^\dagger] = 0 \\
[T^a, Q] &= [T^a, Q^\dagger] = 0
\end{aligned}$$

Here $P^\mu = (H, \vec{\mathbf{P}})$ is the generator of spacetime translations, and T^a are the gauge generators. (This is schematic, with spinor indices suppressed for now. We will restore them later.)

The single-particle states of the theory fall into irreducible representations of this algebra, called **supermultiplets**. Fermion and boson members of a given supermultiplet are **superpartners** of each other. By definition, if $|\Omega\rangle$ and $|\Omega'\rangle$ are superpartners, then $|\Omega'\rangle$ is equal to some combination of $Q, Q^\dagger$ acting on $|\Omega\rangle$.

Therefore, since P^2 and T^a commute with $Q, Q^\dagger$, all members of a given supermultiplet must have the same (mass)^2 and gauge quantum numbers.

Each supermultiplet contains equal numbers of fermions and bosons

Proof: Consider the operator $(-1)^{2S}$ where S is spin angular momentum. Then

$$(-1)^{2S} = \begin{cases} -1 & \text{acting on fermions} \\ +1 & \text{acting on bosons} \end{cases}$$

So, $(-1)^{2S}$ must <u>anticommute</u> with Q and $Q^\dagger$. Now consider all states $|i\rangle$ in a given supermultiplet with the same momentum eigenvalue $p^\mu \neq 0$. These form a complete set of states, so $\sum_j |j\rangle\langle j| = 1$. Now do a little calculation:

$$\begin{aligned}
p^\mu \mathrm{Tr}[(-1)^{2S}] = \sum_i \langle i|(-1)^{2S} P^\mu |i\rangle \quad &= \quad \sum_i \langle i|(-1)^{2S} Q Q^\dagger |i\rangle + \sum_i \langle i|(-1)^{2S} Q^\dagger Q|i\rangle \\
&= \quad \sum_i \langle i|(-1)^{2S} Q Q^\dagger |i\rangle + \sum_i \sum_j \langle i|(-1)^{2s} Q^\dagger |j\rangle\langle j|Q|i\rangle \\
&= \quad \sum_i \langle i|(-1)^{2S} Q Q^\dagger |i\rangle + \sum_j \langle j|Q(-1)^{2S} Q^\dagger |j\rangle \\
&= \quad \sum_i \langle i|(-1)^{2S} Q Q^\dagger |i\rangle - \sum_j \langle j|(-1)^{2S} Q Q^\dagger |j\rangle \\
&= \quad 0.
\end{aligned}$$

The trace just counts the number of boson minus the number of fermion degrees of freedom in the supermultiplet. Therefore, $p^\mu(n_B - n_F) = 0$.

Types of supermultiplets

Chiral (or "Scalar" or "Matter" or "Wess-Zumino") supermultiplet:

1 two-component Weyl fermion, helicity $\pm\frac{1}{2}$. ($n_F = 2$)

2 real spin-0 scalars = 1 complex scalar. ($n_B = 2$)

The Standard Model quarks, leptons and Higgs bosons must fit into these.

Gauge (or "Vector") supermultiplet:

1 two-component Weyl fermion gaugino, helicity $\pm\frac{1}{2}$. ($n_F = 2$)

1 real spin-1 massless gauge vector boson. ($n_B = 2$)

The Standard Model $\gamma, Z, W^{\pm}, g$ must fit into these.

Gravitational supermultiplet:

1 two-component Weyl fermion gravitino, helicity $\pm\frac{3}{2}$. ($n_F = 2$)

1 real spin-2 massless graviton. ($n_B = 2$)

How do the Standard Model quarks and leptons fit in?

Each quark or charged lepton is 1 Dirac = 2 Weyl fermions

Electron: $\Psi_e = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$ $\begin{matrix} \longleftarrow \text{ two-component Weyl LH fermion} \\ \longleftarrow \text{ two-component Weyl RH fermion} \end{matrix}$

Each of e_L and e_R is part of a chiral supermultiplet, so each has a complex, spin-0 superpartner, called $\widetilde{e}_L$ and $\widetilde{e}_R$ respectively. They are called the "left-handed selectron" and "right-handed selectron", although they carry no spin.

The conjugate of a right-handed Weyl spinor is a left-handed Weyl spinor. Define two-component left-handed Weyl fields: $e \equiv e_L$ and $\bar{e} \equiv e_R^\dagger$. So, there are two left-handed chiral supermultiplets for the electron:

$$(e,\ \widetilde{e}_L) \quad \text{and} \quad (\bar{e},\ \widetilde{e}_R^*).$$

The other charged leptons and quarks are similar. We do not need ν_R in the Standard Model, so there is only one neutrino chiral supermultiplet for each family:

$$(\nu_e,\ \widetilde{\nu}_e).$$

Chiral supermultiplets of the Minimal Supersymmetric Standard Model (MSSM):

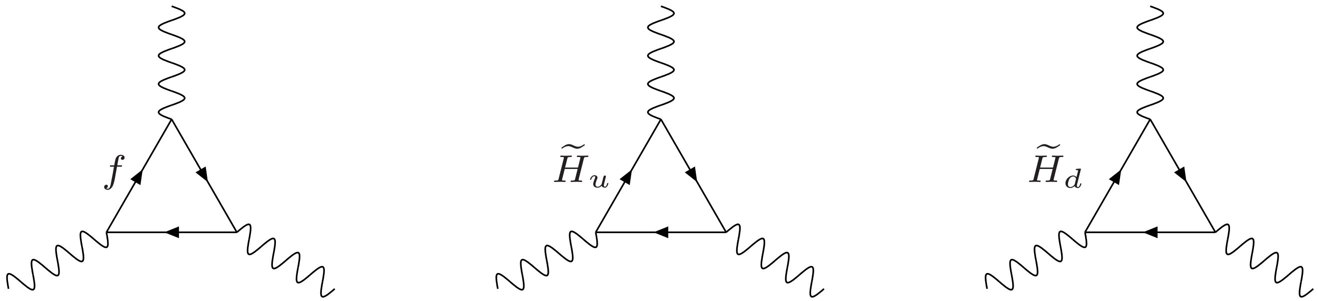
Names		spin 0	spin 1/2	$SU(3)_C,\ SU(2)_L,\ U(1)_Y$
squarks, quarks	Q	$(\widetilde{u}_L \quad \widetilde{d}_L)$	$(u_L \quad d_L)$	$(\,\mathbf{3},\ \mathbf{2}\,,\ \frac{1}{6})$
($\times 3$ families)	$\bar{u}$	$\widetilde{u}^*_R$	$u^\dagger_R$	$(\,\overline{\mathbf{3}},\ \mathbf{1},\ -\frac{2}{3})$
	$\bar{d}$	$\widetilde{d}^*_R$	$d^\dagger_R$	$(\,\overline{\mathbf{3}},\ \mathbf{1},\ \frac{1}{3})$
sleptons, leptons	L	$(\widetilde{\nu} \quad \widetilde{e}_L)$	$(\nu \quad e_L)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ -\frac{1}{2})$
($\times 3$ families)	$\bar{e}$	$\widetilde{e}^*_R$	$e^\dagger_R$	$(\,\mathbf{1},\ \mathbf{1},\ 1)$
Higgs, higgsinos	H_u	$(H^+_u \quad H^0_u)$	$(\widetilde{H}^+_u \quad \widetilde{H}^0_u)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ +\frac{1}{2})$
	H_d	$(H^0_d \quad H^-_d)$	$(\widetilde{H}^0_d \quad \widetilde{H}^-_d)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ -\frac{1}{2})$

The superpartners of the Standard Model particles are written with a $\widetilde{\ }$. The scalar names are obtained by putting an “s” in front, so they are generically called **squarks** and **sleptons**, short for “scalar quark” and “scalar lepton”.

The Standard Model Higgs boson requires two different chiral supermultiplets, H_u and H_d. The fermionic partners of the Higgs scalar fields are called **higgsinos**. There are two charged and two neutral Weyl fermion higgsino degrees of freedom.

Why do we need two Higgs supermultiplets? Two reasons:

1) Anomaly Cancellation



$$\sum_{\text{SM fermions}} Y_f^3 = 0 \qquad + \; 2\left(\frac{1}{2}\right)^3 \qquad + \; 2\left(-\frac{1}{2}\right)^3 \quad = \quad 0$$

This anomaly cancellation occurs if and only if **both** $\widetilde{H}_u$ and $\widetilde{H}_d$ higgsinos are present. Otherwise, the electroweak gauge symmetry would not be allowed!

2) Quark and Lepton masses

Only the H_u Higgs scalar can give masses to charge $+2/3$ quarks (top).
Only the H_d Higgs scalar can give masses to charge $-1/3$ quarks (bottom) and the charged leptons. We will show this later.

The vector bosons of the Standard Model live in gauge supermultiplets:

Names	spin 1/2	spin 1	$SU(3)_C,\ SU(2)_L,\ U(1)_Y$
gluino, gluon	$\widetilde{g}$	g	$(\ \mathbf{8},\ \mathbf{1}\ ,\ 0)$
winos, W bosons	$\widetilde{W}^\pm\quad \widetilde{W}^0$	$W^\pm\quad W^0$	$(\ \mathbf{1},\ \mathbf{3}\ ,\ 0)$
bino, B boson	$\widetilde{B}^0$	B^0	$(\ \mathbf{1},\ \mathbf{1}\ ,\ 0)$

The spin-$1/2$ **gauginos** transform as the adjoint representation of the gauge group. Each gaugino carries a $\widetilde{\ }$. The color-octet superpartner of the gluon is called the **gluino**. The $SU(2)_L$ gauginos are called **winos**, and the $U(1)_Y$ gaugino is called the **bino**.

However, the winos and the bino are not mass eigenstate particles; they mix with each other and with the higgsinos of the same charge.

Recall that if supersymmetry were an exact symmetry, then superpartners would have to be exactly degenerate with each other. For example,

$$m_{\tilde{e}_L} = m_{\tilde{e}_R} = m_e = 0.511 \text{ GeV}$$

$$m_{\tilde{u}_L} = m_{\tilde{u}_R} = m_u$$

$$m_{\tilde{g}} = m_{\text{gluon}} = 0 + \text{QCD-scale effects}$$

etc.

But new particles with these properties have been ruled out long ago, so:

Supersymmetry must be broken in the vacuum state chosen by Nature.

Supersymmetry is thought to be spontaneously broken and therefore hidden, the same way that the full electroweak symmetry $SU(2)_L \times U(1)_Y$ is hidden from very low-energy experiments.

For a clue as to the nature of SUSY breaking, return to our motivation in the Hierarchy Problem. The Higgs mass parameter gets corrections from each chiral supermultiplet:

$$\Delta m_H^2 = \frac{1}{16\pi^2}(\lambda_S - \lambda_F^2)M_{\rm UV}^2 + \ldots$$

The corresponding formula for Higgsinos has no term proportional to $M_{\rm UV}^2$; fermion masses always diverge at worst like $\ln(M_{\rm UV})$. Therefore, if supersymmetry were exact and unbroken, it must be that:

$$\lambda_S = \lambda_F^2,$$

in other words, the dimensionless (scalar)^4 couplings are the squares of the (scalar)-(fermion)-(antifermion) couplings.

If we want SUSY to be a solution to the hierarchy problem, we must demand that this is still true even after SUSY is broken:

The breaking of supersymmetry must be "soft". This means that it does not change the dimensionless terms in the Lagrangian.

The effective Lagrangian of the MSSM is therefore:

$$\mathcal{L} = \mathcal{L}_{\text{SUSY}} + \mathcal{L}_{\text{soft}}$$

- $\mathcal{L}_{\text{SUSY}}$ contains all of the gauge, Yukawa, and dimensionless scalar couplings, and preserves exact supersymmetry
- $\mathcal{L}_{\text{soft}}$ violates supersymmetry, and contains only mass terms and couplings with *positive* mass dimension.

If m_{soft} is the largest mass scale in $\mathcal{L}_{\text{soft}}$, then by dimensional analysis,

$$\Delta m_H^2 = m_{\text{soft}}^2 \left[\frac{\lambda}{16\pi^2} \ln(M_{\text{UV}}/m_{\text{soft}}) + \ldots \right],$$

where λ stands for dimensionless couplings. This is because Δm_H^2 must vanish in the limit $m_{\text{soft}} \to 0$, in which SUSY is restored. Therefore, we expect that m_{soft} should not be much larger than roughly 1000 GeV.

This is the best reason to be optimistic that SUSY will be discovered at the Fermilab Tevatron or the CERN Large Hadron Collider in the near future.

Without further justification, soft SUSY breaking might seem like a rather arbitrary requirement. Fortunately, it arises naturally from the spontaneous breaking of theories with exact SUSY.

Is there any good reason why the superpartners of the Standard Model particles should be heavy enough to have avoided discovery so far? Yes!

- All of the particles in the MSSM that have been discovered as of 1995 (quarks, leptons, gauge bosons) would be exactly massless if the electroweak symmetry were not broken. So their masses are expected to be at most of order $v = 175$ GeV, the electroweak breaking scale. **They are required to be light.**

- All of the particles in the MSSM that have **not** yet been discovered as of 2008 (squarks, sleptons, gauginos, Higgsinos, Higgs scalars) can get a mass even without electroweak symmetry breaking. **They are not required to be light.**

Notations for two-component (Weyl) fermions

Left-handed (LH) two-component Weyl spinor: $\psi_\alpha \quad \alpha = 1, 2$
Right-handed (RH) two-component Weyl spinor: $\psi^\dagger_{\dot\alpha} \quad \dot\alpha = 1, 2$

The Hermitian conjugate of a left-handed Weyl spinor is a right-handed Weyl spinor, and vice versa:

$$(\psi_\alpha)^\dagger = (\psi^\dagger)_{\dot\alpha} \equiv \psi^\dagger_{\dot\alpha}$$

Therefore, **all** spin-$1/2$ fermionic degrees of freedom in any theory can be defined in terms of a list of left-handed Weyl spinors, $\psi_{i\alpha}$ where i is a flavor index. With this convention, right-handed Weyl spinors always carry a dagger: $\psi^{\dagger i}_{\dot\alpha}$.

Products of spinors are defined as:

$$\psi\xi \equiv \psi_\alpha \xi_\beta \epsilon^{\beta\alpha} \qquad \text{and} \qquad \psi^\dagger \xi^\dagger \equiv \psi^\dagger_{\dot\alpha} \xi^\dagger_{\dot\beta} \epsilon^{\dot\alpha\dot\beta}$$

Since ψ and ξ are anti-commuting fields, the antisymmetry of $\epsilon^{\alpha\beta}$ implies:

$$\psi\xi = \xi\psi = (\psi^\dagger\xi^\dagger)^* = (\xi^\dagger\psi^\dagger)^*.$$

To make Lorentz-covariant quantities, define matrices $(\overline{\sigma}_\mu)^{\dot\alpha\beta}$ and $(\sigma_\mu)_{\alpha\dot\beta}$ with:

$$\overline{\sigma}_0 = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \overline{\sigma}_n = -\sigma_n = (\vec{\sigma})_n \quad \text{(for } n = 1, 2, 3\text{)}.$$

Then the Lagrangian for an arbitrary collection of LH Weyl fermions ψ_i is:

$$\mathcal{L} = -i\psi^{\dagger i}\overline{\sigma}^\mu D_\mu \psi_i - \tfrac{1}{2} M^{ij}\psi_i\psi_j - \tfrac{1}{2} M_{ij}\psi^{\dagger i}\psi^{\dagger j}$$

where D_μ = covariant derivative, and the mass matrix M^{ij} is symmetric, with $M_{ij} \equiv (M^{ij})^*$.

Two LH Weyl spinors ξ, χ can form a 4-component Dirac or Majorana spinor:

$$\Psi = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}$$

In the 4-component formalism, the Dirac Lagrangian is:

$$\mathcal{L} \quad = \quad -i\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m\overline{\Psi}\Psi, \qquad \text{where} \qquad \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix},$$

In the two-component fermion language, with spinor indices suppressed:

$$\mathcal{L} = -i\xi^\dagger \overline{\sigma}^\mu \partial_\mu \xi - i\chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi - m(\xi\chi + \xi^\dagger \chi^\dagger),$$

up to a total derivative.

A Majorana fermion can be described in 4-component language in the same way by identifying $\chi = \xi$, and multiplying the Lagrangian by a factor of $\frac{1}{2}$ to compensate for the redundancy.

For example, to describe the Standard Model fermions in 2-component notation:

$$\begin{aligned}\mathcal{L} &= -iQ^{\dagger i}\overline{\sigma}^\mu D_\mu Q_i - i\bar{u}^{\dagger i}\overline{\sigma}^\mu D_\mu \bar{u}_i - i\bar{d}^{\dagger i}\overline{\sigma}^\mu D_\mu \bar{d}_i \\ &\quad -iL^{\dagger i}\overline{\sigma}^\mu D_\mu L_i - i\bar{e}^{\dagger i}\overline{\sigma}^\mu D_\mu \bar{e}_i\end{aligned}$$

with the family index $i = 1, 2, 3$ summed over, color and weak isospin and spinor indices suppressed, and D_μ the appropriate Standard Model covariant derivative, for example,

$$\begin{aligned}D_\mu L &= \left[\partial_\mu + i\frac{g}{2}W_\mu^a \tau^a - i\frac{g'}{2}B_\mu\right]\begin{pmatrix}\nu_e \\ e\end{pmatrix} \\ D_\mu \overline{e} &= \left[\partial_\mu + ig' B_\mu\right]\bar{e}\end{aligned}$$

with τ^a ($a = 1, 2, 3$) equal to the Pauli matrices, and the gauge eigenstate weak bosons are related to the mass eigenstates by

$$\begin{aligned}W_\mu^\pm &= (W_\mu^1 \mp W_\mu^2)/\sqrt{2}, \\ \begin{pmatrix}Z_\mu \\ A_\mu\end{pmatrix} &= \begin{pmatrix}\cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W\end{pmatrix}\begin{pmatrix}W_\mu^3 \\ B_\mu\end{pmatrix}.\end{aligned}$$

Two-component spinor language is much more natural and convenient for SUSY, because the supermultiplets are in one-to-one correspondence with the LH Weyl fermions.

More generally, two-component spinor language is more natural for any theory of physics beyond the Standard Model, because it is an Essential Truth that parity is violated. Nature does not treat left-handed and right-handed fermions the same, and the higher we go in energy, the more essential this becomes.

The simplest SUSY model: a free chiral supermultiplet

The minimum particle content for a SUSY theory is a complex scalar ϕ and its superpartner fermion ψ. We must at least have kinetic terms for each, so:

$$S = \int d^4x \, (\mathcal{L}_{\rm scalar} + \mathcal{L}_{\rm fermion})$$

$$\mathcal{L}_{\rm scalar} = -\partial^\mu \phi^* \partial_\mu \phi \qquad\qquad \mathcal{L}_{\rm fermion} = -i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi$$

A SUSY transformation should turn ϕ into ψ, so try:

$$\delta\phi = \epsilon\psi; \qquad\qquad \delta\phi^* = \epsilon^\dagger \psi^\dagger$$

where $\epsilon =$ infinitesimal, anticommuting, constant spinor, with dimension $[\text{mass}]^{-1/2}$, that parameterizes the SUSY transformation. Then we find:

$$\delta\mathcal{L}_{\rm scalar} = -\epsilon\partial^\mu\psi\partial_\mu\phi^* - \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi.$$

We would like for this to be canceled by an appropriate SUSY transformation of the fermion field. . .

To have any chance, $\delta\psi$ should be linear in $\epsilon^\dagger$ and in ϕ, and must contain one spacetime derivative. There is only one possibility, up to a multiplicative constant:

$$\delta\psi_\alpha = i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu\phi; \qquad\qquad \delta\psi^\dagger_{\dot\alpha} = -i(\epsilon\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^*$$

With this guess, one obtains:

$$\delta\mathcal{L}_{\rm fermion} = -\delta\mathcal{L}_{\rm scalar} + \big(\text{total derivative}\big)$$

so the action S is indeed invariant under the SUSY transformation, justifying the guess of the multiplicative factor. This is called the free Wess-Zumino model.

Furthermore, if we take the commutator of two SUSY transformations:

$$\delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) = i(\epsilon_1\sigma^\mu\epsilon_2 - \epsilon_2\sigma^\mu\epsilon_1)\partial_\mu\phi$$

Since ∂_μ corresponds to the spacetime 4-momentum P_μ, this has exactly the form demanded by the SUSY algebra discussed earlier. (More on this soon.)

The fact that two SUSY transformations give back another symmetry (namely a spacetime translation) means that the SUSY algebra "closes".

If we do the same check for the fermion ψ:

$$\begin{aligned}\delta_{\epsilon_2}(\delta_{\epsilon_1}\psi_\alpha) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\psi_\alpha) \quad &= \quad i(\epsilon_1\sigma^\mu\epsilon_2 - \epsilon_2\sigma^\mu\epsilon_1)\partial_\mu\psi_\alpha \\ &\qquad -i\epsilon_{1\alpha}(\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi) + i\epsilon_{2\alpha}(\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi)\end{aligned}$$

The first line is expected, but the second line only vanishes on-shell (when the classical equations of motion are satisfied). This seems like a problem, since we want SUSY to be a valid symmetry of the quantum theory (off-shell)!

To show that there is no problem, we introduce another bosonic spin-0 field, F, called an auxiliary field. Its Lagrangian density is:

$$\mathcal{L}_{\text{aux}} = F^* F$$

Note that F has no kinetic term, and has dimensions [mass]^2, unlike an ordinary scalar field. It has the not-very-exciting equations of motion $F = F^* = 0$.

The auxiliary field F does not affect the dynamics, classically or in the quantum theory. But it does appear in modified SUSY transformation laws:

$$\begin{aligned} \delta\phi &= \epsilon\psi \\ \delta\psi_\alpha &= i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu\phi + \epsilon_\alpha F \\ \delta F &= i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi \end{aligned}$$

Now the total Lagrangian

$$\mathcal{L} = -\partial^\mu\phi^*\partial_\mu\phi - i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + F^*F$$

is still invariant, and also one can now check:

$$\delta_{\epsilon_2}(\delta_{\epsilon_1}X) - \delta_{\epsilon_1}(\delta_{\epsilon_2}X) \;=\; i(\epsilon_1\sigma^\mu\epsilon_2 - \epsilon_2\sigma^\mu\epsilon_1)\partial_\mu X$$

for each of $X = \phi, \phi^*, \psi, \psi^\dagger, F, F^*$, without using equations of motion.

So in the “modified” theory, SUSY does close off-shell as well as on-shell.

The auxiliary field F is really just a book-keeping device to make this simple. We can see why it is needed by considering the number of degrees of freedom on-shell (classically) and off-shell (quantum mechanically):

	ϕ	ψ	F
on-shell ($n_B = n_F = 2$)	2	2	0
off-shell ($n_B = n_F = 4$)	2	4	2

(Going on-shell eliminates half of the propagating degrees of freedom of the fermion, because the Lagrangian density is linear in time derivatives, so that the fermionic canonical momenta are not independent phase-space variables.)

The auxiliary field will also plays an important role when we add interactions to the theory, and in gaining a simple understanding of SUSY breaking.

Noether's Theorem tells us that for every symmetry, there is a conserved current, and SUSY is not an exception. The **supercurrent** J^μ_α is an anti-commuting 4-vector that also carries a spinor index.

By the usual Noether procedure, one finds for the supercurrent (and its conjugate $J^\dagger$), in terms of the variations of the fields δX for $X = (\phi, \phi^*, \psi, \psi^\dagger, F, F^*)$:

$$\epsilon J^\mu + \epsilon^\dagger J^{\dagger\mu} \quad \equiv \quad \sum_X \delta X \, \frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} - K^\mu,$$

where K^μ satisfies $\delta\mathcal{L} = \partial_\mu K^\mu$. One finds:

$$J^\mu_\alpha = (\sigma^\nu \overline{\sigma}^\mu \psi)_\alpha \, \partial_\nu \phi^*; \qquad\qquad J^{\dagger\mu}_{\dot\alpha} = (\psi^\dagger \overline{\sigma}^\mu \sigma^\nu)_{\dot\alpha} \, \partial_\nu \phi.$$

The supercurrent and its hermitian conjugate are separately conserved:

$$\partial_\mu J^\mu_\alpha = 0; \qquad\qquad \partial_\mu J^{\dagger\mu}_{\dot\alpha} = 0,$$

as can be verified by use of the equations of motion.

From the conserved supercurrents one can construct the conserved charges:

$$Q_\alpha = \sqrt{2}\int d^3x\, J^0_\alpha; \qquad\qquad Q^\dagger_{\dot\alpha} = \sqrt{2}\int d^3x\, J^{\dagger 0}_{\dot\alpha},$$

As quantum mechanical operators, they satisfy:

$$\left[\epsilon Q + \epsilon^\dagger Q^\dagger, X\right] = -i\sqrt{2}\,\delta X$$

for any field X. Let us also introduce the 4-momentum operator $P^\mu = (H, \vec{P})$, which satisfies:

$$[P_\mu, X] = i\partial_\mu X.$$

Now by using the canonical commutation relations of the fields, one finds:

$$\begin{aligned}\left[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger,\, \epsilon_1 Q + \epsilon_1^\dagger Q^\dagger\right] &= 2(\epsilon_2\sigma_\mu\epsilon_1^\dagger - \epsilon_1\sigma_\mu\epsilon_2^\dagger)\, P^\mu \\ \left[\epsilon Q + \epsilon^\dagger Q^\dagger,\, P\right] &= 0\end{aligned}$$

This implies. . .

The SUSY Algebra

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu,$$
$$\{Q_\alpha, Q_\beta\} = \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} = 0$$
$$[Q_\alpha, P^\mu] = [Q^\dagger_{\dot\alpha}, P^\mu] = 0$$

This time in non-schematic form, with the spinor indices and the factors of 2 in their proper places.

(The commutators turned into anti-commutators in the first two, when we extracted the anti-commutating spinors ϵ_1, ϵ_2.)

Masses and Interactions for Chiral Supermultiplets

The Lagrangian describing a collection of free, massless, chiral supermultiplets is

$$\mathcal{L} = -\partial^\mu \phi^{*i} \partial_\mu \phi_i - i\psi^{\dagger i} \overline{\sigma}^\mu \partial_\mu \psi_i + F^{*i} F_i.$$

Question: How do we make mass terms and interactions for these fields, while still preserving supersymmetry invariance?

Answer: choose a **superpotential**,

$$W = \tfrac{1}{2} M^{ij} \phi_i \phi_j + \tfrac{1}{6} y^{ijk} \phi_i \phi_j \phi_k.$$

It does not depend on ϕ^{*i}, only the ϕ_i. It must be an analytic function of the scalar fields treated as complex variables.

The superpotential W contains masses M^{ij} and couplings y^{ijk}, which must be symmetric under interchange of i, j, k.

Supersymmetry is very restrictive; you cannot just do anything you want!

The resulting Lagrangian for interacting chiral supermultiplets is:

$$\begin{aligned}\mathcal{L} &= -\partial^\mu \phi^{*i} \partial_\mu \phi_i - i\psi^{\dagger i}\overline{\sigma}^\mu \partial_\mu \psi_i \\ &\quad -\tfrac{1}{2}\left(M^{ij}\psi_i\psi_j + y^{ijk}\phi_i\psi_j\psi_k\right) + \text{c.c.} \\ &\quad -V(\phi_i, \phi^{*i})\end{aligned}$$

where the scalar potential is:

$$\begin{aligned}V(\phi_i, \phi^{*i}) &= M_{ik}M^{kj}\phi^{*i}\phi_j + \frac{1}{2}M^{in}y_{jkn}\phi_i\phi^{*j}\phi^{*k} \\ &\quad + \frac{1}{2}M_{in}y^{jkn}\phi^{*i}\phi_j\phi_k + \frac{1}{4}y^{ijn}y_{kln}\phi_i\phi_j\phi^{*k}\phi^{*l}\end{aligned}$$

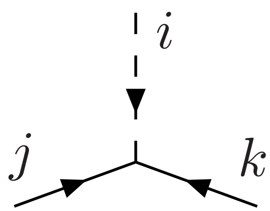
The superpotential W "encodes" all of the information about these masses and interactions.

The superpotential $W = M^{ij}\phi_i\phi_j + y^{ijk}\phi_i\phi_j\phi_k$ determines all non-gauge masses and interactions.

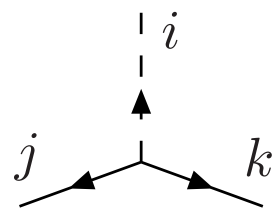
Both scalars and fermions have squared mass matrix $M_{ik}M^{kj}$.

The interaction Feynman rules for the chiral supermultiplets are:

Yukawa interactions:

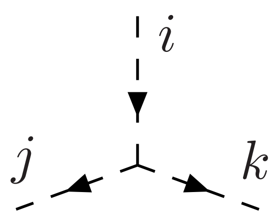


$-iy^{ijk}$

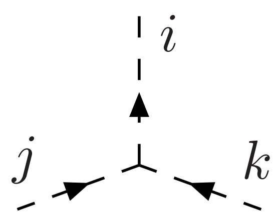


$-iy_{ijk}$

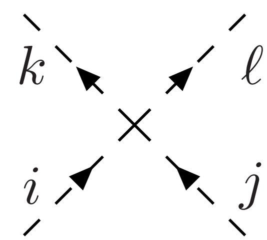
Scalar interactions:



$-iM^{in}y_{njk}$



$-iM_{in}y^{njk}$



$-iy^{ijn}y_{k\ell n}$

Covered in Lecture 1:

- The Hierarchy Problem, $m_Z \ll m_{\rm Planck}$, is a strong motivation for supersymmetry (SUSY)
- In SUSY, all particles fall into:
  - Chiral supermultiplet $=$ complex scalar boson and fermion partner
  - Gauge supermultiplet $=$ vector boson and gaugino fermion partner
  - Gravitational supermultiplet $=$ graviton and gravitino fermion partner
- The Minimal Supersymmetric Standard Model (MSSM) introduces squarks, sleptons, Higgsinos, gauginos as the superpartners of Standard Model states
- Two-component fermion notation: ψ_α $=$ LH fermion, $\psi^\dagger_{\dot\alpha}$ $=$ RH fermion
- The Wess-Zumino Model Lagrangian describes a single chiral supermultiplet
- The Supersymmetry Algebra
- Superpotentials and interactions